

HAMILTONIAN CYCLES IN CAYLEY GRAPHS OF IMPRIMITIVE COMPLEX REFLECTION GROUPS

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ABSTRACT. Generalizing a result of Conway, Sloane, and Wilkes for real reflection groups, we show the Cayley graph of an imprimitive complex reflection group with respect to standard generating reflections has a Hamiltonian cycle. This is consistent with the long-standing conjecture that for every finite group, G , and every set of generators, S , of G the undirected Cayley graph of G with respect to S has a Hamiltonian cycle.

1. INTRODUCTION

For a finite group G and a subset S of $G \setminus \{1\}$, the (right, undirected) *Cayley graph* of G with respect to S , $\Gamma(G, S)$, has vertices corresponding to the elements $g \in G$ and edges (g, gs) and (g, gs^{-1}) for each $g \in G$ and $s \in S$. The Cayley graph is vertex-transitive, regular, and connected when S generates G , which we assume throughout. Label the edge from g to gs by s and that from gs to g by s^{-1} (note that edge labels for $\Gamma(G, S)$ are drawn from $S \cup S^{-1}$). It is common to consider both together as a single undirected edge with s and s^{-1} indicating travel along the edge in the appropriate direction.

A *path* in Γ is an ordered sequence of adjacent vertices in Γ and a path is *self-avoiding* if no vertex appears more than once. A *Hamiltonian path* is a self-avoiding path containing every vertex of Γ . When the initial and final vertex of a Hamiltonian path are adjacent it determines a *Hamiltonian cycle* and a graph containing a Hamiltonian cycle is called *Hamiltonian*.

The question, dating back to 1969 in a monograph by Lovász, of whether every connected vertex-transitive graph has a Hamiltonian path, remains unresolved. The stronger claim, that every connected vertex-transitive graph has a Hamiltonian cycle, is known to be false and it has been observed that the four known counterexamples are not Cayley graphs. The resulting conjecture that for every finite group G and any generating set S the Cayley graph $\Gamma(G, S)$ has a Hamiltonian cycle also remains unresolved and finding such a cycle is an NP-complete problem in general. See [WG84, CG96, KM09, PR09] for surveys of the status and history of the problem and references, including those supplying counter-conjectures. When S is not closed under inversion, it is possible for the directed graph with vertices the elements of G and only edges (g, gs) for $g \in G$ and $s \in S$ to have no Hamiltonian cycle. For instance, the directed circulant graph on \mathbb{Z}_{12} with generators 3, 4 (and 6) is not Hamiltonian (see [WG84] and [LW99]).

The conjecture that every (undirected) Cayley graph is Hamiltonian is easy to prove for abelian groups and known to be true for several specific types of groups that are nearly abelian, with either specific or arbitrary generating sets. For instance the conjecture has been shown true when

- G is a p -group [Wit86],
- the commutator subgroup G' is a cyclic p -group [Mar83, Dur83, Dur85, KW85, DGMW98],
- the order of G has few prime factors [KMM⁺12],
- the order of G is odd and G' has order pq or is cyclic of order $p^a q^b$ for $a, b \geq 0$ [WM],
- G is nilpotent and G' is cyclic [GWM14].

It is also known that the Cayley graph of the semidirect product of two cyclic groups with respect to a specific generating set is Hamiltonian [Als89]. We prove the conjecture is true for the infinite family of imprimitive complex reflection groups, which are highly non-abelian semidirect products involving the symmetric group, with respect to specific commonly used generating sets of reflections. Specifically we prove the following, where μ is the cyclic group of de -th roots of unity.

Theorem 1.1. *Let $G = G(de, e, n) \simeq \mu^n \rtimes S_n$ be an irreducible imprimitive complex reflection group and S be a standard generating set for G . The (undirected right) Cayley graph $\Gamma(G, S)$ has a Hamiltonian cycle.*

Our result generalizes that in [CSW89], which provides an algorithm to generate a Hamiltonian cycle in each $\Gamma(G, S)$ where G is a finite real reflection group and S is the standard set of generating simple reflections. That paper utilized the Coxeter presentation of the groups to give an inductive proof of the existence, and hence recursive construction, of a Hamiltonian cycle. It also explicitly treats the small number of base cases.

Although there is no such uniformly well-behaved presentation or set of generators for complex reflection groups (see [BM98, Shi02]), we use those that go back to [Cox67, Coh76] and are given in the standard references [BMR98, LT09]. While these presentations, generating sets, and resulting Cayley graphs do not tend to satisfy the usual conditions for the existence of Hamiltonian cycles currently given in the literature (see further discussion at the end of Section 2), they do allow for an inductive approach similar to that in [CSW89]. In order to exploit that approach we must treat six infinite families of groups as base cases. For three of these families we explicitly write down Hamiltonian cycles and in the remaining three cases our proofs provide a method for doing so.

In two cases we utilize a process we call flipping which, when applied to a self-avoiding path, produces a new self-avoiding path. This is similar to a process utilized in a probabilistic algorithm to find Hamiltonian cycles in general graphs (see [AV79]). It would be interesting to further explore the application of the flipping process and related algorithms to Cayley graphs of complex reflection groups and in particular, to determine whether there exist obstructions to their success in these graphs and if so, under what conditions.

The paper is organized as follows. In Section 2, following [GM06], we review necessary facts about real reflection groups and summarize the classification of complex reflection groups due to Shephard and Todd [ST54]. The classification consists of a three-parameter infinite family, $G(de, e, n)$, along with 34 exceptional groups. This paper treats only the $G(de, e, n)$, though we have conducted some initial investigations for the exceptional complex reflection groups. Generating sets S for the $G(de, e, n)$ are also given in Section 2. Our formulations of the commonly used Factor Group Lemma (see [KMM⁺12]), the flipping process, and the method of lifting cycles from quotient graphs are described in Section 3. The main result, that all $\Gamma(G(de, e, n), S)$ are Hamiltonian, and its proof, including all base case lemmas, appear in Section 4.

2. BACKGROUND ON REFLECTION GROUPS

A *Coxeter system* (G, S) is a group G with set of generators $S = \{s_1, \dots, s_n\}$ that has a presentation of the form

$$G = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} \rangle,$$

where $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$ for $1 \leq i \neq j \leq n$. Such a group is called a *Coxeter group* and is more commonly denoted W due to the connection with Weyl groups. The presentations of Coxeter groups are classified using diagrams that graphically encode the s_i and the m_{ij} . The classification of finite irreducible Coxeter groups consists of four infinite families and six exceptional

groups. Finite Coxeter groups have a geometric incarnation as they are exactly the finite groups generated by orthogonal reflections of a real vector space (see [Bou02, Hum90]). It can also be shown that there is a natural set of generating reflections up to conjugacy, so such a choice is fixed and these are termed *simple reflections*. Every Cayley graph $\Gamma(G, S)$ with G a finite irreducible Coxeter group and S a set of simple reflections is shown in [CSW89] to have a Hamiltonian cycle.

The notions of reflection and classification of finite groups generated by reflections extend to the setting of an n -dimensional complex vector space V (for a brief survey, see [GM06]). A linear transformation $r : V \rightarrow V$ is a *reflection* if it is of finite order and has a $+1$ -eigenspace of dimension $n - 1$. In the remaining complex dimension the reflection acts by a root of unity and hence may have order greater than two. A finite subgroup, G , of $\text{GL}(V)$ generated by reflections is called a *reflection group on V* . Since G is finite, the standard averaging technique makes it possible to fix a non-degenerate G -invariant hermitian form on V and consider G as a subgroup of the unitary group on V . Finiteness of G also guarantees the representation on V is completely reducible, which means it suffices to consider reflection groups and spaces on which they act irreducibly. More precisely, G is said to act irreducibly in dimension k if its fixed point space is of dimension $n - k$ and it acts irreducibly when restricted to the complement of that fixed point space.

We describe an infinite family of complex reflection groups. Let $d, e, n \geq 1$, let μ denote the cyclic group of de -th roots of unity, and let ζ generate μ . Under the *standard monomial representation*, $G(de, e, n)$ consists of

$$\begin{aligned} &\text{monomial matrices with nonzero entries } \zeta^{a_1}, \dots, \zeta^{a_n}, \\ &\text{such that } (\zeta^{a_1} \dots \zeta^{a_n})^d = 1, \text{ or equivalently, } a_1 + \dots + a_n \equiv 0 \pmod{e}. \end{aligned}$$

Each such monomial matrix may be written as a product of a diagonal matrix with entries $\zeta^{a_1}, \dots, \zeta^{a_n}$ and a permutation matrix (obtained by permuting columns of the identity matrix). This provides an alternative description of $G(de, e, n)$ as an index e subgroup of $\mu \wr S_n = \mu^n \rtimes S_n$, and makes it clear that $|G(de, e, n)| = d^n e^{n-1} n!$. In this perspective, $G(de, e, n)$ consists of all

$$(a_1, \dots, a_n \mid \sigma) \text{ such that } a_i \in \mathbb{Z}_{de}, a_1 + \dots + a_n \equiv 0 \pmod{e}, \text{ and } \sigma \in S_n,$$

and the action of S_n on μ^n providing the semidirect product structure on $G(de, e, n)$ is

$$\sigma.(a_1, \dots, a_n) = (a_{\sigma(1)}, \dots, a_{\sigma(n)}),$$

so that if $\sigma, \tau \in S_n$,

$$(a_1, \dots, a_n \mid \sigma)(b_1, \dots, b_n \mid \tau) = (a_1 + b_{\sigma(1)}, \dots, a_n + b_{\sigma(n)} \mid \sigma\tau),$$

where $\sigma\tau$ is computed by applying σ first, τ second. This is consistent with the use of the right Cayley graph and with matrix multiplication where $(a_1, a_2, \dots, a_n \mid \sigma)$ represents the $n \times n$ matrix whose only nonzero entries are the ζ^{a_i} in position $i, \sigma(i)$ for $1 \leq i \leq n$.

A reflection group G on V is *imprimitive* if there is a decomposition $V = V_1 \oplus V_2 \oplus \dots \oplus V_k$ into proper nonzero subspaces such that G permutes the subspaces. The groups $G(de, e, n)$ with $de, n \geq 2$ are imprimitive in their action on a system of lines orthogonal to the reflecting hyperplanes and the exceptional groups are primitive. The symmetric groups $G(1, 1, n)$ do not act irreducibly in the standard monomial representation but do act irreducibly on the complement of the span of the sum of all the basis vectors and are primitive on this $(n - 1)$ -dimensional subspace.

The irreducible finite reflection groups were classified in [ST54] (see also [LT09]) and consist of the

- groups $G(de, e, n)$, with $de \geq 2$, $n \geq 1$, and $(de, e, n) \neq (2, 2, 2)$, which are imprimitive and irreducible in dimension n ,
- symmetric groups $G(1, 1, n)$, which are primitive and irreducible in dimension $n - 1$, and
- 34 primitive exceptional groups, numbered G_4, \dots, G_{37} , irreducible in dimensions 2 to 8.

The reason for the numbering is that the original classification listed $G(d, 1, 1)$, $G(1, 1, n)$ and $G(de, e, n)$ with $de, n \geq 2$ separately. If there is a G -invariant real subspace, V_0 , of V so that the canonical map $\mathbb{C} \otimes_{\mathbb{R}} V_0 \rightarrow V$ is a bijection, then G is a *real reflection group*. The finite real reflection groups occur in the classification from [ST54] as:

- $G(1, 1, n)$ of type A_{n-1} , the symmetric group S_n ,
- $G(2, 1, n)$ of type B_n , the binary octahedral group $\{\pm 1\}^n \rtimes S_n$,
- $G(2, 2, n)$ of type D_n , an index two subgroup of $\{\pm 1\}^n \rtimes S_n$,
- $G(m, m, 2)$ of type $I_2(m)$, the dihedral group of order $2n$, and
- $G_{23}, G_{28}, G_{30}, G_{35}, G_{36}$, and G_{37} of types H_3, F_4, H_4, E_6, E_7 , and E_8 respectively.

The following explicit choices of representations of generators for the groups are provided in [LT09] and are consistent with the presentations provided by the diagrams in [BMR98]. For $1 \leq i < n$, let

$$r_i = (0, \dots, 0 \mid (i \ i+1)) = \text{the identity matrix with columns } i \text{ and } i+1 \text{ interchanged,}$$

$$s = (-1, 1, 0, \dots, 0 \mid (1 \ 2)) = \begin{bmatrix} 0 & \zeta^{-1} & 0 & \dots & 0 \\ \zeta & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}, \text{ and}$$

$$t = (e, 0, \dots, 0 \mid 1) = \begin{bmatrix} \zeta^e & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}.$$

Then, using $e = 1$ in t for $G(d, 1, n)$,

$$\begin{aligned} G(d, 1, n) &= \langle t, r_1, r_2, \dots, r_{n-1} \rangle \text{ for } d \geq 2, n \geq 1, \\ G(e, e, n) &= \langle s, r_1, r_2, \dots, r_{n-1} \rangle, \text{ for } e \geq 2, n \geq 1, \text{ and} \\ G(de, e, n) &= \langle s, t, r_1, r_2, \dots, r_{n-1} \rangle, \text{ for } d, e, n \geq 2. \end{aligned}$$

In particular, $G(d, 1, n)$ and $G(e, e, n)$ are generated by n reflections (are *well-generated*) while the groups $G(de, e, n)$ with $d, e, n \geq 2$ require $n+1$ generators.

We note that the groups $G(de, e, n)$ do not tend to satisfy the sufficient conditions for the existence of Hamiltonian cycles currently known in the literature. For example, Theorem 1.2 of [KMM⁺12] and related references guarantee the existence of a Hamiltonian cycle in any Cayley graph of a finite group whose order has a “small” prime factorization of certain forms, and those of [Wit86, WM] apply to groups of prime power and odd order respectively. But such results will not apply in general to the $G(de, e, n)$. Since $G(de, e, n)$ is a semidirect product involving S_n , rather than cyclic groups of prime order and abelian groups, it is not of the form addressed in [Dur83] or [Als89], and it is also not nilpotent for $n \geq 3$, so the results of [GWM14] on nilpotent groups do not generally apply. The results of [DGMW98, KW85] apply to groups with commutator subgroups that are cyclic of prime power order. Using the presentations and the number of one-dimensional characters of S_n and μ , it is easy to count one-dimensional characters of $G = G(de, e, n)$ and the abelian group G/G' and thus to compute $|G'|$. This yields $|G'| = d^{n-1}$ when $e = n = 1$, $|G'| = (de)^{n-1}n!/4$ when $n = 2$ and e is even, and $|G'| = (de)^{n-1}n!/2$ in the remaining cases. This makes it clear that the results of [DGMW98, KW85] do not generally apply.

Some Hamiltonicity results address groups with simple types of generators and relations. For instance, [GM07] addresses groups generated by an involution and an element of order at least three whose product is of order three, while the three lemmas in Section 2 of [PR09] involve generators that are involutions and/or satisfy very simple relations. The generating sets we use typically do not meet these hypotheses on the number of, orders of, or relations between the generators. The main result in [PR09] does provide, for an arbitrary group, the existence of a Hamiltonian cycle in its Cayley graph with respect to a relatively small generating set. While striking, this result serves a different purpose than we address here. Lastly, several results in the broader graph theory literature on existence of Hamiltonian cycles treat graphs with sufficiently large degree and/or sufficiently small connectivity (see e.g., [Dir52, Jac80]), which do not apply to the families of Cayley graphs we consider.

3. LIFTING CYCLES FROM QUOTIENT GRAPHS

Rather than specify a sequence of vertices, an equivalent way to describe a Hamiltonian cycle is to specify an initial vertex v_1 and a sequence of edges. In the case of a Cayley graph it suffices to use $v_1 = 1$ and to list the sequence of edges as a sequence of generators, which we denote below with square brackets to connote an ordered list. If $P = [s_1, s_2, \dots, s_{k-1}, s_k]$ denotes the sequence of edges for a path, then $P^\#$ will denote $[s_1, s_2, \dots, s_{k-1}]$.

Our treatment of the base cases for the inductive proof of the main result for $G(de, e, n)$ involves the construction of explicit Hamiltonian cycles. We make use of the following techniques.

Lemma 3.1. *Let G be a group with generating set S and let $\Gamma = \Gamma(G, S)$ denote its Cayley graph. Let $B = [s_1, s_2, \dots, s_k]$, where each $s_i \in S$. Let $v = s_1 s_2 \dots s_k$. Let $w_0 = 1$ and for $0 < i < k$, let $w_i = s_1 s_2 \dots s_i$. If $n > 0$, the path P defined by $B^\#$ is self-avoiding if and only if whenever $0 \leq m < n$ and $v^m w_i = w_j$, it follows that $m = 0$ and $i = j$.*

Proof. The forward direction is immediate. To establish the converse, observe that vertices along the path P are given by expressions of the form $v^m w_i$, where $0 \leq m < n$ and $0 \leq i < k$. Suppose that for two such vertices we have $v^{m_1} w_i = v^{m_2} w_j$. We can assume that $m_1 \leq m_2$. Cancellation yields $v^m w_i = w_j$, where $m = m_2 - m_1 < n$. It follows that $m_1 = m_2$ and $i = j$, establishing that P is self-avoiding. \square

Remark 3.2. *Note that, in the context of Lemma 3.1, $B^\#$ is a Hamiltonian path if it is self-avoiding and the size of the group G is nk . The path B^n is closed if v is order n . When B^n is a Hamiltonian cycle, the condition*

$$v^m w_i = w_j \text{ implies } m = 0 \text{ and } i = j$$

is equivalent to

$$w_0, w_1, \dots, w_{k-1} \text{ visits each left coset of } H = \langle v \rangle \text{ exactly once.}$$

See Lemma 2.6 in [KMM⁺12].

If a Hamiltonian path is known it may be possible to alter the path into a cycle with a process we call *flipping*. This is similar to a process used in an algorithm given in [AV79]. See Figure 1.

Definition 3.3. If $P = [s_1, s_2, \dots, s_n]$ is a path and for some $k < n$ the vertices $s_1 s_2 \dots s_k$ and $s_1 s_2 \dots s_n$ are adjacent via an edge labeled s , then the *flip* of P with respect to s , $F(P, s)$, is the path $[s_1, s_2, \dots, s_k, s, s_n^{-1}, s_{n-1}^{-1}, \dots, s_{k+2}^{-1}]$.

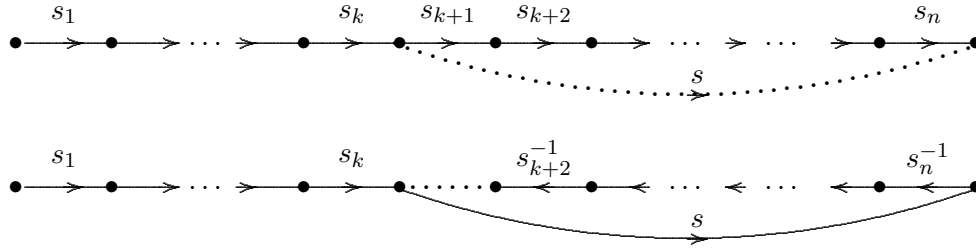


FIGURE 1. Flipping a path

The walks P and $F(P, s)$ visit exactly the same vertices. In particular, if P is self-avoiding, then $F(P, s)$ is self-avoiding.

Several of the proofs presented below make use of a lifting technique similar to that presented in the proof of Theorem 1.1 in [CSW89]. It relies on a method for combining disjoint cycles in the Cayley graph.

Definition 3.4. If C_1 and C_2 are disjoint cycles in $\Gamma(G, S)$ and $r \in S$, we say that C_1 and C_2 satisfy the *commutative joining property* with respect to r if there exist edges $(g_i, g_i s)$ in the C_i and such that $g_2 = g_1 r$ and r commutes with s .

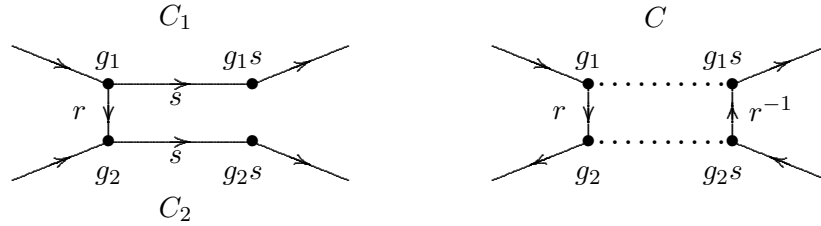


FIGURE 2. Combining cycles with the commutative joining property

When two cycles satisfy the commutative joining property they can be combined into a new cycle C which spans the union of the vertex sets of C_1 and C_2 . This is accomplished by first taking the edge $(g_1, g_1 r)$ from C_1 to C_2 , traversing C_2 in reverse until arriving at $g_2 s$, then taking the edge $(g_2 s, g_2 s r^{-1})$. Because r and s commute, $g_2 s r^{-1} = g_1 r s r^{-1} = g_1 s$ and the cycle is completed by continuing on around C_1 to finish back at g_1 . See Figure 2.

The following propositions extend this joining strategy to sets of disjoint cycles.

Proposition 3.5. Let G be a finite group with generating set S . Let $\{C, C_1, \dots, C_n\}$ be a collection of disjoint cycles in $\Gamma(G, S)$ and $r \in S$. If C and each C_l have the commutative joining property with respect to r then all of the cycles can be combined to form a single cycle spanning the union of the various vertex sets.

Proof. For each cycle C_l , find distinct edge pairs $(g_l, g_l s_l)$ in C and $(g_l r, g_l s_l r)$ in C_l where r and s_l commute. When C is combined with one of the other cycles as described above, the resulting cycle contains each of the edges $(g_l, g_l s_l)$ corresponding to the remaining C_l . Therefore, the process can be iterated until the cycle contains all of the C_l . \square

Proposition 3.6. Let G be a finite group with generating set $S = \{r_1, \dots, r_n\}$, where r_n is of order two. Let $H = \langle r_1, \dots, r_{n-1} \rangle$. If $\Gamma(H, S \setminus \{r_n\})$ has a Hamiltonian cycle C_H with the property

whenever (g, gs) , (gs, gst) are consecutive edges in C_H , either s or t must commute with r_n , then $\Gamma(G, S)$ has a Hamiltonian cycle.

Proof. Let $\{H_l \mid l = 1, \dots, k\}$ be the set of left cosets of H with $H_1 = H$. Let Γ_l denote the subgraph of $\Gamma(G, S \setminus \{r_n\})$ with vertex set H_l . Each Γ_l is isomorphic to $\Gamma(H, S \setminus \{r_n\})$. Since $\Gamma(G, S)$ is connected, the set of subgraphs Γ_l can be linked using edges labelled r_n . To start, set $C = C_H$.

Inductive hypothesis: $\Gamma(G, S)$ contains a cycle C such that

- the vertex set of C is the union of some subcollection of the left cosets of H , and
- if r_i and r_j are consecutive edge labels in C and neither is r_n then r_i and r_j also appear as consecutive edge labels in the cycle C_H .

The inductive hypothesis is trivially satisfied when $C = C_H$. If the vertex set of C is all of G we are finished. If not, there is an index l such that C and Γ_l are disjoint and linked by r_n . Find consecutive edges $(g', g'r_i)$ and $(g'r_i, g'r_i r_j) = (g, gr_j)$ in C and a vertex $g_l \in \Gamma_l$ with $gr_n = g_l$. Observe that neither r_i nor r_j is r_n since r_n is of order 2.

From the inductive hypothesis the edge labels r_i and r_j must appear as consecutive edge labels in C_H . It follows that either r_i or r_j commutes with r_n . Assume that r_n and r_j commute; the case where r_n commutes with r_i is handled in similar fashion.

The cycle C_H must contain an edge of the form (h, hr_j) for some $h \in H$. It follows that the cycle $C_l = g_l h^{-1} C_H$ is a spanning cycle for Γ_l and contains the edge $(g_l, g_l r_j)$. Since C contains $(g_l, g_l r_j)$ and $gr_n = g_l$, we conclude that C and C_l have the commutative joining property with respect to r_n and can be combined as described above, creating a new cycle containing all of the vertices from C and C_l , which we again call C .

The new vertex set C is a union of cosets of H . Consecutive edge labels in C , neither of which is r_n , signify consecutive edges coming from either C_l or the previous version of C and, in either case, must appear as consecutive edge labels in C_H . The inductive hypothesis remains valid.

Continuing in this manner we arrive at a cycle whose vertex set is the union of all of the left cosets of H and is, therefore, a Hamiltonian cycle in $\Gamma(G, S)$. \square

In the preceding proof, any choice of an edge labeled r_n connecting C and Γ_l will work in the inductive step. In general, there will be many such edges. If the conditions on C_H are relaxed to allow the cycle to contain instances of consecutive edge labels signifying generators, neither of which commute with r_n , then the inductive step fails unless the edge connecting C and Γ_l can be chosen to avoid these “bad” portions of the copies of C_H that have accumulated in C .

Definition 3.7. Let G be a finite group with generating set S . Let C be a cycle in $\Gamma(G, S)$. For r in S define the *badness* of C with respect to r , denoted $bad(C, r)$, to be the number of instances in C of consecutive edge labels s and t where r fails to commute with both of the generators s and t .

Corollary 3.8. Let G be a finite group with generating set $S = \{r_1, \dots, r_n\}$, where r_n is of order two. Let $H = \langle r_1, \dots, r_{n-1} \rangle$. Let $\{\Gamma_l\}$ denote the components of $\Gamma(G, S \setminus \{r_n\})$. Denote by $c(i, j)$ the number of distinct edges in $\Gamma(G, S)$ that are labeled r_n and connect Γ_i and Γ_j . If $\Gamma(H, S \setminus \{r_n\})$ contains a spanning cycle C_H with $bad(C_H, r_n) < c(i, j)$ for all positive $c(i, j)$ then $\Gamma(G, S)$ contains a Hamiltonian cycle.

Proof. The proof proceeds exactly as the proof of Proposition 3.6. Each constructed C_l has the same sequence of edge labels as C_H so that $bad(C_l, r_n) = bad(C_H, r_n)$. In the inductive step, when a Γ_l is chosen that is connected to C by an edge labeled r_n , there must be $c(i, l)$ different vertices in C , coming from some previously incorporated $C_i \subset \Gamma_i$, that connect C to Γ_l by applying r_n . The edges in C that pass through these vertices come from C_i and bear consecutive edge labels

consistent with the sequence of edge labels in C_H . Since $c(i, l)$ exceeds $\text{bad}(C, r_n)$, a suitable choice of connecting edge can be made. \square

4. APPLICATION TO CAYLEY GRAPHS OF COMPLEX REFLECTION GROUPS

Examining the diagrams and corresponding presentations in [BMR98] or the matrix forms of the generators given in Section 2 shows that in

$$\begin{aligned} &G(d, 1, n) \text{ with } d \geq 2, n \geq 3, \\ &G(e, e, n) \text{ with } e \geq 2, n \geq 4, \text{ and} \\ &G(de, e, n) \text{ with } d, e \geq 2, n \geq 4, \end{aligned}$$

r_{n-1} commutes with all generators other than r_{n-2} . Thus Proposition 3.6 will apply to treat the induction step in the proof of Theorem 4.7 for these cases. It will be necessary to separately address the remaining infinite base cases,

$$\begin{aligned} &G(d, 1, 2) \text{ with } d \geq 2, \\ &G(e, e, 3) \text{ with } e \geq 2, \\ &G(de, e, 2) \text{ with } d, e \geq 2, \text{ and} \\ &G(de, e, 3) \text{ with } d, e \geq 2. \end{aligned}$$

In [BMR98] the case of $e = 2$ in $G(de, e, n)$ is treated separately (though some indication was given it could be combined, we never resolved an issue concerning the double braid relation). Here we treat separately the cases of $d = 2$ in $G(de, e, 2)$ and $G(de, e, 3)$ but do not distinguish $e = 2$. Explicit Hamiltonian cycles are given for the families $G(d, 1, 2)$, $G(2e, e, 2)$, and $G(e, e, 3)$ in Lemmas 4.1, 4.2, and 4.3. In Lemma 4.4 we construct a Hamiltonian cycle in the graph for $G(de, e, 2)$ by applying the flipping process to an explicit Hamiltonian path. Lemmas 4.5 and 4.6 achieve the lifting of the cycles given for $G(de, e, 2)$ and $G(2e, e, 2)$ to $G(de, e, 3)$ and $G(2e, e, 3)$ respectively.

Recall that we are using right Cayley graphs and all arithmetic is resolved with the appropriate modulus. In the first two lemmas denote r_1 by r and in all proofs denote the elements of the permutation group S_3 by:

$$\sigma_0 = 1, \quad \sigma_1 = (1 \ 2), \quad \sigma_2 = (2 \ 3), \quad \sigma_3 = (1 \ 3), \quad \sigma_4 = (1 \ 2 \ 3), \quad \sigma_5 = (1 \ 3 \ 2).$$

Lemma 4.1. *Let $G = G(d, 1, 2)$ with $d \geq 2$, which has presentation*

$$G = \langle t, r \mid t^d = r^2 = 1, trtr = rtrt \rangle.$$

If $B = [[t]^{d-1}, r]$, then B^{2d} is a Hamiltonian cycle in $\Gamma(G, \{r, t\})$.

Proof. The elements of $G(d, 1, 2)$ are of the form

$$\{(a, b \mid \sigma) \mid 0 \leq a, b < d \text{ and } \sigma \in S_2 = \{\sigma_0, \sigma_1\}\}.$$

Entries in the tuples representing vertices are determined up to congruence modulo d . Denote the walk $B^{2d\#}$ by P . The number of vertices visited by P is $2d^2$ which is the order of the group $G(d, 1, 2)$. It suffices to show that B^{2d} is closed and P is self-avoiding.

Let $v = t^{d-1}r$ and $w_i = t^i$ for $0 \leq i < d$. Since t is of order d , the vertices in w_0, \dots, w_{d-1} are distinct. In tuple notation, t is $(1, 0 \mid \sigma_0)$ and r is $(0, 0 \mid \sigma_1)$. From this,

$$v = t^{d-1}r = (d-1, 0 \mid \sigma_0)(0, 0 \mid \sigma_1) = (-1, 0 \mid \sigma_1),$$

$v^2 = (-1, -1 \mid \sigma_0)$, and v is of order $2d$, establishing that B^{2d} is closed.

Suppose $0 \leq m < 2d$, $0 \leq i, j < d$ and $v^m w_i = w_j$. Since $w_i = (i, 0 \mid \sigma_0)$ and $w_j = (j, 0 \mid \sigma_0)$, m must be even, say $m = 2k$. Then $v^m w_i = w_j$ becomes $(-k + i, -k \mid \sigma_0) = (j, 0 \mid \sigma_0)$. It follows that $k \equiv 0 \pmod{d}$ and $i \equiv j \pmod{d}$. Since $0 \leq i, j, k < d$ it must be that $k = 0$ and $i = j$. Lemma 3.1 applies to allow us to conclude that $P = B^{2d\#}$ is self-avoiding. \square

Lemma 4.2. *Let $G = G(2e, e, 2)$ with $e \geq 3$, which has presentation*

$$G = \langle r, s, t \mid r^2 = s^2 = t^2 = 1, tsr = srt, \underbrace{rtsrsrs \cdots}_{e+1 \text{ factors}} = \underbrace{tsrsrs \cdots}_{e+1 \text{ factors}} \rangle.$$

If $B = [[[r, s]^{e-1}], r, t]$ then the walk B^4 is a Hamiltonian cycle in $\Gamma(G, \{r, s, t\})$.

Proof. The elements of $G(2e, e, 2)$ are of the form

$$\{(a, b \mid \sigma) \mid 0 \leq a, b < 2e, a + b \equiv 0 \pmod{e}, \text{ and } \sigma \in S_2 = \{\sigma_0, \sigma_1\}\}.$$

Entries in the tuples representing vertices are determined up to congruence modulo $2e$. We show that B^4 is closed and $P = B^{4\#}$ is self-avoiding.

Let $w_{2i} = (rs)^i$ and $w_{2i+1} = (rs)^i r$ for $0 \leq i < e$. Let $v = (rs)^{e-1} rt = (e-1, 1 \mid \sigma_1)$. It follows that $v^2 = (e, e \mid \sigma_0)$ so that v is of order 4 and B^4 is closed. The number of vertices visited by P is $8e$ which is the order of the group $G(2e, e, 2)$. It suffices to show that P is self-avoiding, which we establish using Lemma 3.1.

Let $0 \leq m < 4$ and suppose we have $v^m w_i = w_j$. We may assume that j is even, otherwise, right-multiply by r . It follows that the resulting m and i must have the same parity. There are two cases to consider:

Case 1: m is even. Set $m = 2l$ for $l = 0$ or 1 and $i = 2l_i$, $j = 2l_j$. Then $v^m w_i = w_j$ becomes

$$(le + l_i, le - l_i \mid \sigma_0) = (l_j, -l_j \mid \sigma_0).$$

This yields a system of two congruences that imply $l_i - l_j \equiv 0 \pmod{e}$, forcing $l_i = l_j$ and $i = j$. It follows that l must be 0 and so $m = 0$.

Case 2: m is odd. Set $m = 2l + 1$ for $l = 0$ or 1 and $i = 2l_i + 1$, $j = 2l_j + 1$. Then $v^m w_i = w_j$ becomes

$$((l+1)e - l_i - 1, le + l_i + 1 \mid \sigma_0) = (l_j, -l_j \mid \sigma_0)$$

yielding the system of congruences

$$\begin{aligned} e + le - l_i - 1 &\equiv l_j \pmod{2e} \\ le + l_i + 1 &\equiv -l_j \pmod{2e} \end{aligned}$$

which has no solution. Lemma 3.1 applies to allow us to conclude that P is self-avoiding. \square

In the next lemma, to avoid subscripts, continue to denote r_1 by r and denote r_2 by q .

Lemma 4.3. *Let $G = G(e, e, 3)$ with $e \geq 2$, which has presentation*

$$G = \langle s, r, q \mid s^2 = r^2 = q^2 = 1, sqs = qsq, rqr = qrq, qsrqsr = srqsrq, \underbrace{srs \cdots}_e \text{ factors} = \underbrace{rsr \cdots}_e \text{ factors} \rangle$$

Let $A = [q, s, q, r, q, r]$ and $B = [A^{e\#}, s]$. Then B^e is a Hamiltonian cycle in $\Gamma = \Gamma(G, \{s, r, q\})$.

Proof. The elements of $G(e, e, 3)$ are of the form

$$\{(a, b, c \mid \sigma) \mid 0 \leq a, b, c < e, a + b + c \equiv 0 \pmod{e}, \text{ and } \sigma \in S_3 = \{\sigma_0, \sigma_1, \dots, \sigma_5\}\}.$$

Entries in the tuples are determined up to congruence modulo e .

Let $1 = w_0, w_1, \dots, w_{k-1}$ denote the vertices, in sequence, of the walk $A^{e\#}$ and let $v = w_{k-1}s$. Note that $k = 6e$. In tuple notation,

$$\begin{aligned} q &= (0, 0, 0 \mid \sigma_2), & qsqr &= (-1, 0, 1 \mid \sigma_5), \\ qs &= (-1, 0, 1 \mid \sigma_4), & qsqrq &= (-1, 0, 1 \mid \sigma_1), \\ qsq &= (-1, 0, 1 \mid \sigma_3), & qsqrqr &= (-1, 0, 1 \mid \sigma_0). \end{aligned}$$

Note that all $\sigma \in S_3$ appear in this list.

The walk determined by $A^{e\#}s$ ends in the vertex $v = (qsqrqr)^e s = (1, -1, 0 \mid \sigma_0)$ and v has order e . Suppose $v^m w_i = w_j$. Write $w_i = (qsqrqr)^{l_i} v_i$ and $w_j = (qsqrqr)^{l_j} v_j$ where $v_i, v_j \in \{1, q, qs, qsq, qsqr, qsqrq\}$. It follows that the tuples for v_i and v_j must contain the same σ and consequently must be equal. Cancellation results in a reduction to either

$$v^m (qsqrqr)^l = 1 \text{ or } v^m = (qsqrqr)^l$$

where $l = |l_i - l_j|$.

The first equation asserts that $(m - l, -m, l \mid \sigma_0) = (0, 0, 0 \mid \sigma_0)$ forcing $m = l = 0$ and $w_i = w_j$. The second equation asserts that $(m, -m, 0 \mid \sigma_0) = (-l, 0, l \mid \sigma_0)$, again forcing $m = l = 0$ and $w_i = w_j$. Lemma 3.1 allows us to conclude that P is self-avoiding. The order of the group G is $6e^2$ and v is order e so B^e is closed and is a Hamiltonian cycle. \square

Lemma 4.4. *Let $G = G(de, e, 2)$ with $d \geq 3, e \geq 2$, which has presentation*

$$G = \langle t, s, r \mid r^2 = s^2 = t^d = 1, tsr = srt, \underbrace{rtsrsrs \cdots}_{e+1 \text{ factors}} = \underbrace{tsrsrs \cdots}_{e+1 \text{ factors}} \rangle.$$

Let $A = [[t]^{d-1}, s]^{2d}$, $B = [A\#, r]$ and $S = \{t, s, r\}$. Then $B^{e\#}$ is a Hamiltonian path in $\Gamma(G, S)$ and two successive flips of this path with respect to s produce a Hamiltonian path P such that $[P, r]$ is a Hamiltonian cycle in $\Gamma(G, S)$.

Proof. Let $H = \langle s, t \rangle$. An elementary argument establishes that the vertex set of the walk A is H . The proof of Lemma 4.1 can be modified slightly to provide an argument showing that A is a Hamiltonian cycle in $\Gamma(H, \{s, t\})$. It follows that $|H| = 2d^2$ and H has index e in G .

The right cosets of H are $H, Hsr, H(sr)^2, \dots, H(sr)^{e-1}$. To see this, suppose for some $0 \leq i, j < e$, that $(sr)^i$ and $(sr)^j$ determine the same right coset of H . Assume $i \geq j$. It follows that $(sr)^k \in H$ where $k = i - j$. Elements of H of the form $(a, b \mid \sigma_0)$ must arise as $(t^{d-1}s)^{2m}t^n$ for suitable m, n . But $(sr)^k = (-k, k \mid \sigma_0)$ while $(t^{d-1}s)^{2m}t^n = (-me + ne, -me \mid \sigma_0)$. If $(sr)^k = (t^{d-1}s)^{2m}t^n$ then $k \equiv -me \pmod{de}$ and hence k must be congruent to 0 modulo e , but since $0 \leq k < e$, it must be that $k = 0$.

The graph $\Gamma(G, S)$ is partitioned into e subgraphs corresponding to the cosets of H . Each of these subgraphs has a spanning cycle obtained by applying A starting at each of $1, sr, (sr)^2, \dots, (sr)^{e-1}$. If the final s in each of these cycles is replaced by r the effect is the same as the concatenation of A with $[s, r]$, effectively moving to the next coset. Since s is order 2, $[A, s, r] = [A\#, r] = B$. Thus, $B^{e\#}$ utilizes each subgraph cycle to form a Hamiltonian path in $\Gamma(G, S)$.

The Hamiltonian path $B^{e\#}$ ends at the vertex $(-e, e \mid \sigma_1)$. It remains to show how to alter this path so that it ends in a vertex adjacent to 1. The cosets of H are closed with respect to right

multiplication by s so that flipping the path $B^{e\#}$ with respect to s only alters the segment spanning the last coset of H . The first flip using s reverses the cycle (path) in the last coset, and ends at

$$(-(e-1), e-1 \mid \sigma_0)t = (1, e-1 \mid \sigma_0).$$

This reversal changes each t to t^{-1} in the portion of the path that is traversed backwards. The second flip using s produces a path that ends at

$$(1, e-1 \mid \sigma_0)st^{-1} = (0, e \mid \sigma_1)t^{-1} = (0, 0 \mid \sigma_1),$$

so the terminal vertex of the path is now adjacent, via r , to 1. Refer to Figure 3. \square

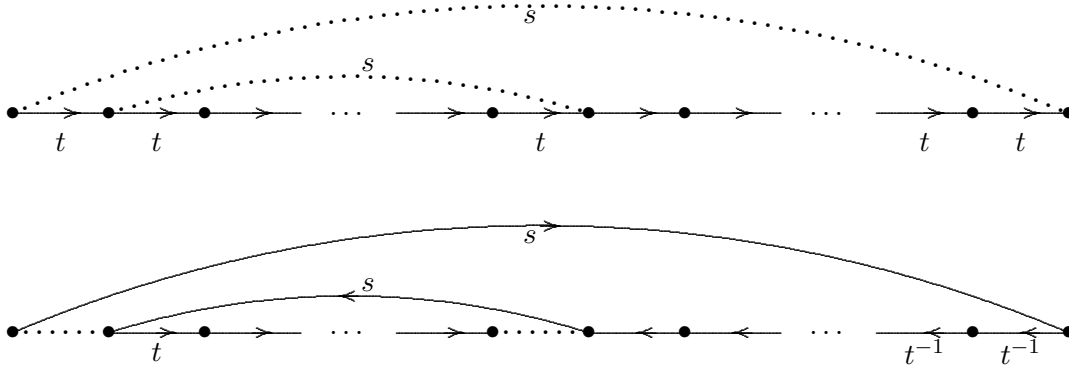


FIGURE 3. Altering the Hamiltonian path in Lemma 4.4

Lemma 4.5. *Let $G = G(de, e, 3)$ with $d \geq 3, e \geq 2$, which has presentation*

$$G = \langle t, s, r_1, r_2 \mid r_1^2 = r_2^2 = s^2 = t^d = 1, tr_2 = r_2t, tsr_1 = sr_1t, sr_2s = r_2sr_2, r_1r_2r_1 = r_2r_1r_2, \underbrace{r_1tsr_1sr_1s \cdots}_{e+1 \text{ factors}} = \underbrace{tsr_1sr_1s \cdots}_{e+1 \text{ factors}}, r_2sr_1r_2sr_1 = sr_1r_2sr_1r_2 \rangle.$$

Then $\Gamma(G, \{t, s, r_1, r_2\})$ has a Hamiltonian cycle.

Proof. Let $H = \langle t, s, r_1 \rangle$ and observe that H is isomorphic to $G(de, e, 2)$. Set $S = \{t, s, r_1, r_2\}$. The components, $\{\Gamma_l\}$, of $\Gamma(G, S \setminus \{r_2\})$ are subgraphs on the various cosets of H with r_2 -labeled edges removed. Since r_2 is of order two, we have a setting consistent with Corollary 3.8.

For the cycle C_H we take the Hamiltonian cycle constructed in Lemma 4.4. Using the notation of Lemma 4.4, r_1 is matched with r . Since t and r_2 commute, the value of $bad(C_H, r_2)$ is found by counting the number of occurrences of consecutive rs or sr edge pairs. No such pairs appeared in the initial Hamiltonian path used in the construction of C_H . At most two such pairs are introduced by the two s -flips and they must occur at the beginning and end of the altered path through the final coset (examine Figure 3). Consequently, $bad(C_H, r_2) \leq 2$.

Suppose Γ_i and Γ_j are connected by an r_2 edge. Let $H_i = g_iH$ and $H_j = g_jH$ denote the left cosets which are the vertex sets of Γ_i and Γ_j , respectively. Then there are $h_i, h_j \in H$ with $g_ih_ir_2 = g_jh_j$. The generators t and r_2 commute so that for $0 \leq k < d$ we have

$$g_ih_it^kr_2 = g_ih_ir_2t^k = g_jh_jt^k$$

and each $(g_ih_it^k, g_jh_jt^k)$ is an edge linking Γ_i and Γ_j . This shows that $c(i, j) \geq d$. Since $d \geq 3$ and $bad(C_H, r_2) \leq 2$, Corollary 3.8 applies and we conclude that $\Gamma(G, \{t, s, r_1, r_2\})$ has a Hamiltonian cycle. \square

Lemma 4.6. *Let $G = G(2e, e, 3)$ with $e \geq 2$, which has presentation*

$$G = \langle t, s, r_1, r_2 \mid r_1^2 = r_2^2 = s^2 = t^2 = 1, tr_2 = r_2t, tsr_1 = sr_1t, sr_2s = r_2sr_2, r_1r_2r_1 = r_2r_1r_2, \\ \underbrace{r_1tsr_1sr_1s \cdots}_{e+1 \text{ factors}} = \underbrace{tsr_1sr_1s \cdots}_{e+1 \text{ factors}}, r_2sr_1r_2sr_1 = sr_1r_2sr_1r_2 \rangle.$$

Then $\Gamma(G, \{t, s, r_1, r_2\})$ has a Hamiltonian cycle.

Proof. Let $S = \{t, s, r_1, r_2\}$ and $H = \langle S \setminus \{r_2\} \rangle$. Observe that each generator in S is of order 2 and that H is isomorphic to $G(2e, e, 2)$. Let $\{\Gamma_l\}$ denote the components of $\Gamma(G, S \setminus \{r_2\})$. Each Γ_l has as vertex set some left coset of H and all are isomorphic to $\Gamma_H = \Gamma(H, S \setminus \{r_2\})$. Let C_H denote the Hamiltonian cycle in Γ_H described in Lemma 4.2. Since $\text{bad}(C_H, r_2)$ is quite large (its value is $8(e-1)$) the procedure used in the proof of Lemma 4.5 will fail. Our approach here involves chaining together self-avoiding paths derived from C_H into a large cycle C which spans $4e$ of the $6e$ coset graphs. This large cycle can then be combined with spanning cycles in the remaining components using the commutative joining property to obtain the desired result.

The left cosets of H are described as

$$\begin{aligned} H(k, 1) &= \{(k, b, c \mid \sigma) \in G, \sigma \in \{\sigma_3, \sigma_5\}\}, \\ H(k, 2) &= \{(a, k, c \mid \sigma) \in G, \sigma \in \{\sigma_2, \sigma_4\}\}, \text{ and} \\ H(k, 3) &= \{(a, b, k \mid \sigma) \in G, \sigma \in \{\sigma_0, \sigma_1\}\}, \end{aligned}$$

where $0 \leq k < 2e$. For example, $H = H(0, 3)$.

From Lemma 4.2, $C_H = [[r_1, s]^{e-1}, r_1, t]^4$. It is easy to verify that $(r_1s)^{e-1}r_1t = ts$. It follows that the path Q obtained from $C_H^\#$ by flipping with respect to s is given by

$$Q = [[r_1, s]^{e-1}, r_1, t, s, [[r_1, s]^{e-1}, r_1, t]^2, [r_1, s]^{e-1}].$$

If Q is applied starting at 1 the ending vertex is tsr_1 . See Figure 4.

From $tsr_1r_2 = (e-1, 1, 0 \mid \sigma_2)$ we see that

$$\begin{aligned} (tsr_1r_2)^{2i} &= (i(2e-2), i, i \mid \sigma_0) \in H(i, 3) \text{ and} \\ (tsr_1r_2)^{2i+1} &= (i(2e-2) + e-1, i+1, i+1 \mid \sigma_2) \in H(i+1, 2). \end{aligned}$$

It follows that tsr_1r_2 has order $4e$ and the cycle $[t, s, r_1, r_2]^{4e}$ visits each coset of the form $H(k, 2)$ or $H(k, 3)$. If each factor of $[t, s, r_1]$ is replaced by Q the result is a cycle C that spans $4e$ of the $6e$ different coset graphs. We use the fact that t and r_2 commute and complete the argument by showing that any component Γ_l disjoint from C contains a spanning cycle C_l such that C and C_l have the commutative joining property with respect to r_2 .

We first establish that for any k with $0 \leq k < 2e$ there is an edge (v, vt) in C with $vr_2 \in H(k, 1)$. To accomplish this we observe that

$$(a, b, c \mid \sigma_1)r_2 = (a, b, c \mid \sigma_5) \text{ and } (a, b, c \mid \sigma_4)r_2 = (a, b, c \mid \sigma_3)$$

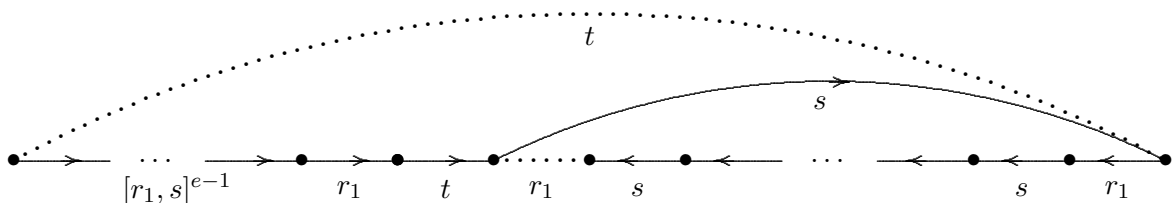


FIGURE 4. Adapting the Hamiltonian cycle from Lemma 4.2

and so it suffices to show that for any k with $0 \leq k < 2e$ there is an edge (v, vt) in C with v of the form $(k, b, c \mid \sigma_1)$ or $(k, b, c \mid \sigma_4)$.

The path Q contains exactly three edges of the form (v, vt) and the vertex v is one of

$$(r_1 s)^{e-1} r_1, \quad (r_1 s)^{e-1} r_1 t s (r_1 s)^{e-1} r_1 \quad \text{or} \quad (r_1 s)^{e-1} r_1 t s (r_1 s)^{e-1} r_1 t (r_1 s)^{e-1} r_1.$$

Only the first two of these vertices are adjacent via r_2 to a vertex in some $H(k, 1)$. Using $(r_1 s)^{e-1} r_1 t = ts$ and the fact that all generators are of order two, the two useful vertices are tst and st . The complete collection of edges (v, vt) in C where vr_2 is not a vertex in C corresponds to the set of vertices of one of the forms $(tsr_1 r_2)^i tst$ or $(tsr_1 r_2)^i st$ for $0 \leq i < 4e$. It is convenient to split into the cases where i is odd and where i is even. For $0 \leq j < 2e$, direct computations produce:

$$\begin{aligned} (tsr_1 r_2)^{2j} tst &= (e - 2j - 1, e + j + 1, j \mid \sigma_1), \\ (tsr_1 r_2)^{2j+1} tst &= (-2j - 2, j + 1, e + j + 1 \mid \sigma_4), \\ (tsr_1 r_2)^{2j} st &= (-2j - 1, e + j + 1, j \mid \sigma_1), \text{ and} \\ (tsr_1 r_2)^{2j+1} st &= (e - 2j - 2, j + 1, e + j + 1 \mid \sigma_4). \end{aligned}$$

The set $\{-2j - 2, -2j - 1 \mid 0 \leq j < 2e\}$ will contain a complete set of residues modulo $2e$ and, consequently, for any k there is an edge (v, vt) in C where $vr_2 \in H(k, 1)$.

We have shown that if Γ_l is a coset graph disjoint from C then there is an edge (v, vt) in C and vr_2 is a vertex in Γ_l . The vertex set of Γ_l is a left coset of H so that the cycle $(vr_2)C_H$ is a spanning cycle in Γ_l . Vertex transitivity can be used to alter this cycle to obtain a spanning cycle C_l in Γ_l containing the edge $(vr_2, vr_2 t)$. It follows that C and C_l have the commutative joining property with respect to r_2 and Proposition 3.5 applies to complete the proof. \square

At this point, we have established Hamiltonicity in each of the required base cases. The remaining graphs under consideration can be handled by induction.

Theorem 4.7. *Let G be an irreducible imprimitive complex reflection group, i.e., let $G = G(de, e, n)$ with $de \geq 2$, $n \geq 1$, and $(de, e, n) \neq (2, 2, 2)$. For standard generating sets S (as given in Section 2), the undirected right Cayley graph $\Gamma(G, S)$ has a Hamiltonian cycle.*

Proof. When $n = 1$ the group is cyclic so $\Gamma(G, S)$ is circulant and has a Hamiltonian cycle. Lemmas 4.1-4.6 prove the statement for $n = 2, 3$. Proceed by induction on n . The generating set for $G = G(de, e, n)$ being considered can be written as $S \cup \{r_n\}$ where the subgroup H of G generated by S is isomorphic to $G(de, e, n - 1)$, r_n commutes with all but one of the elements of S and that element is of order two. By induction, there is a Hamiltonian cycle C_H in $\Gamma(H, S)$. This cycle can contain no consecutive edge labels both of which fail to commute with r_n , i.e., $\text{bad}(C_H, r_n) = 0$. Thus by Proposition 3.6, Γ_G also contains a Hamiltonian cycle. \square

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